

PROBABILISTIC INTERPRETATION OF ELECTRICAL IMPEDANCE TOMOGRAPHY

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ABSTRACT. In this paper, we give probabilistic interpretations of both, the forward and the inverse problem of electrical impedance tomography with possibly anisotropic, merely measurable conductivities: Using the theory of symmetric Dirichlet spaces, Feynman-Kac type formulae corresponding to different electrode models on bounded Lipschitz domains are derived. Moreover, we give a probabilistic interpretation of the Calderón inverse conductivity problem in terms of reflecting diffusion processes and their corresponding boundary trace processes.

1. INTRODUCTION

In this work we derive purely probabilistic representations in the form of Feynman-Kac type formulae for solutions of the conductivity equation

$$(1) \quad \nabla \cdot (\kappa \nabla u) = 0$$

posed on a bounded, simply connected domain $D \subset \mathbb{R}^d$, $d \geq 2$, with Lipschitz boundary ∂D and possibly anisotropic uniformly elliptic and uniformly bounded conductivity subject to different boundary conditions modeling electrode measurements. Moreover, we provide a probabilistic interpretation of the inverse conductivity problem of electrical impedance tomography (EIT), the so-called *Calderón problem* which reads as follows: *Given the Cauchy data on the boundary, i.e., all pairs of voltage and current measurements on ∂D , is it possible to determine the conductivity κ uniquely?* Our probabilistic interpretation generalizes results for the reflecting Brownian motion (RBM) obtained by Hsu [26].

Although it is beyond the scope of this paper we would like to emphasize that Feynman-Kac type representation formulae provide a versatile tool when it comes to problems with random, rapidly oscillating coefficients. For a one-dimensional statistical inverse problem, such a setting was recently studied by Nolen and Papanicolaou [35]. Moreover, due to the advent of multicore computing architectures, Feynman-Kac type representation formulae can yield fast and scalable parallel algorithms for stochastic numerics, see for instance the recent articles [5, 30, 31, 43].

For non-divergence form operators with smooth coefficients on smooth bounded domains, it is well-known that reflecting diffusion processes satisfy Skorohod type stochastic differential equations, see the celebrated article by Lions and Sznitman [33]. The construction in the case of divergence form operators with merely measurable coefficients requires the theory of symmetric Dirichlet spaces and is a major challenge for an arbitrary Euclidean domain D due to the fact that the underlying Dirichlet space is not necessarily regular on $L^2(D)$. Thus in general, the reflecting diffusion process can only be constructed on some abstract closure of D , see the pioneering work by Chen

[9]. When D is a bounded Lipschitz domain, Bass and Hsu [4] constructed a reflecting Brownian motion living on \overline{D} by showing that the so-called Martin-Kuramochi boundary coincides with the Euclidean boundary in this case. A general diffusion process on a bounded Lipschitz domain, even allowing locally a finite number of Hölder cusps, was first constructed by Fukushima and Tomisaki [20]. We use this process here in order to derive Feynman-Kac type representation formulae for the solutions of boundary value problems with Neumann, respectively Robin boundary conditions modeling electrode measurements, for the conductivity equation (1). However, the construction in [20] concentrates on the strong Feller resolvent of the process rather than on its transition kernel density. Therefore we give a proof of the Hölder continuity up to the boundary of the latter.

Probabilistic approaches to both, parabolic and elliptic boundary value problems for second order differential operators have been studied by many authors, starting with Feynman's Princeton thesis [14] and the article [27] by Kac. The probabilistic approach to the Dirichlet problem for a general class of second-order elliptic operators with merely measurable coefficients, even allowing singularities of a certain type, was elaborated by Chen and Zhang [10]; See also Zhang's paper [44]. However, there are only few works that treat Feynman-Kac type representation formulae for Neumann or Robin type boundary conditions. Moreover the approaches existing in the literature consider either the Laplacian, see e.g. [4, 6, 25], or non-divergence form operators with smooth coefficients, see e.g. [15, 36, 7]. For the particular case of the conductivity equation (1) on bounded domains we generalize both, the Feynman-Kac formula for the Robin problem on domains with boundary of class C^3 for an isotropic $C^{2,\gamma}$ -smooth conductivity, $\gamma > 0$, obtained by Papanicolaou [36] as well as the representation obtained by Benchérif-Madani and Pardoux [7] for the Neumann problem under similar regularity assumptions. While both of the aforementioned approaches use stochastic differential equations and Itô calculus, our approach is based on the theory of symmetric Dirichlet spaces, following the pioneering work by Bass and Hsu [4] for the RBM. Our Feynman-Kac type formula for the Robin boundary condition corresponding to the so-called *complete electrode model* is valid for possibly anisotropic uniformly elliptic uniformly bounded conductivities with merely measurable components on bounded Lipschitz domains. For the Neumann boundary condition corresponding to the so-called *continuum model* we have to impose a slightly stronger regularity assumption for the conductivity in some neighborhood of the boundary.

During the preparation of this work we became aware of the paper [11] by Chen and Zhang, where a probabilistic approach to some mixed boundary value problems with singular coefficients is derived. In contrast to our setting, however, the mixed boundary conditions studied there come from a singular lower-order term of the differential operator.

Finally we would like to point out that our Feynman-Kac type formulae yield continuity of the electric potential up to the boundary, a result that is apparently not so easy to obtain by standard Sobolev regularity theory for linear elliptic boundary value problems. In fact, by the celebrated Wiener criterion, solutions of the Laplace equation with Dirichlet boundary conditions are continuous at a boundary point if and only if the so-called Wiener integral associated with this point diverges. For Robin boundary conditions, on the other hand, the situation was not as well understood for quite a long time. In 2001 Griepentrog and Recke were able to prove continuity up to the boundary under very general assumptions, however, using a rather abstract framework based on

Sobolev-Campanato spaces, cf. [21]. In contrast to their (much more general) method, our proof is purely probabilistic and rather simple.

The rest of the paper is structured as follows: We start in Section 2 by collecting some preliminaries concerning electrical impedance tomography as well as standard Dirichlet space theory for reflecting diffusion processes. In Section 3 we show that the transition kernel density of the underlying reflecting diffusion process is Hölder continuous up to the boundary which enables the refinement of the process. Subsequently, in Section 4, the Feynman-Kac type formulae will be derived. Furthermore, a martingale formulation for the complete electrode model is given. Then in Section 5 we provide a probabilistic interpretation of the Calderón problem. Finally, we conclude with a brief summary of our results.

2. PRELIMINARIES

First a word about notation: We denote by $\langle \cdot, \cdot \rangle$ the standard inner product on $L^2(D)$ and by $\|\cdot\|$ the corresponding norm. We use the subscript ‘ \diamond ’ to denote standard Lebesgue, respectively Sobolev spaces with a certain normalization, namely

$$L^2_\diamond(\partial D) := \left\{ \phi \in L^2(\partial D) : \int_{\partial D} \phi \, d\sigma(x) = 0 \right\},$$

where σ denotes the $(d-1)$ -dimensional Lebesgue surface measure, and

$$H^1_\diamond(D) := \left\{ \phi \in H^1(D) : \langle \phi, 1 \rangle = 0 \right\}.$$

For the reason of notational compactness we use the Iverson brackets: Let S be a mathematical statement, then

$$[S] = \begin{cases} 1, & \text{if } S \text{ is true} \\ 0, & \text{otherwise.} \end{cases}$$

We also use the Iverson brackets $[x \in B]$ to denote the indicator function of a set B , which we abbreviate by $[B]$ if there is no danger of confusion. In what follows, various unimportant constants will be denoted c, c_1, c_2, \dots and they may vary from line to line.

2.1. Modeling of electrode measurements in EIT. Throughout this paper, let D denote a bounded Lipschitz domain with connected complement and Lipschitz parameters (r_D, c_D) , i.e., there exist constants $r_D > 0$ and $c_D > 0$ so that for every $x \in \partial D$ there is a ball $B(x, r_D)$ such that after rotation and translation $\partial D \cap B(x, r_D)$ is the graph of a Lipschitz function in the first $d-1$ coordinates with Lipschitz constant no larger than c_D and $D \cap B(x, r_D)$ lies above the graph of this function. Note that without loss of generality we may take $c_D > 1$. We assume that the, possibly anisotropic, conductivity is defined by a symmetric, matrix-valued function $\kappa : D \rightarrow \mathbb{R}^{d \times d}$ with components in $L^\infty(D)$ such that κ is uniformly bounded and uniformly elliptic, i.e., there exists some constant $c_0 > 0$ such that

$$(2) \quad c_0^{-1} \|\xi\|^2 \leq \xi \cdot \kappa(x) \xi \leq c_0 \|\xi\|^2, \quad \text{for every } \xi \in \mathbb{R}^d \text{ and a.e. } x \in D.$$

Moreover we will explicitly state if we use one of the following assumptions:

- (A1) There exists a neighborhood \mathcal{U} of the boundary ∂D such that $\kappa \mathcal{U}$ is isotropic and equal to 1.
- (A2) There exists a finite collection $\Gamma = \{\Gamma_j, 0 \leq j \leq M\}$ of $C^{1,1}$ surfaces that divide D into disjoint open sub-domains $\{\mathcal{U}_j, 0 \leq j \leq M\}$ so that $\partial D \subset \partial \mathcal{U}_0$ and $\kappa \mathcal{U}_j \in W^{1,\infty}(\mathcal{U}_j)$, $j = 1, \dots, M$.

We are going to require assumption (A1) for the probabilistic interpretation of the Neumann boundary value problem corresponding to the continuum model in Section 4 while assumption (A2) will only be used for the probabilistic interpretation of the inverse conductivity problem in Section 5. In particular the derivation of the Feynman-Kac type formula for the Robin boundary condition corresponding to the complete electrode model does not require any of these additional regularity assumptions.

Remark 1. Notice that the assumption (A1) from above is not very restrictive as it can be shown using extension techniques that for domains $\hat{D}, D \subset \mathbb{R}^d$ such that $D \subset \hat{D}$, the knowledge of both, the Dirichlet-to-Neumann map Λ_κ on ∂D and $\kappa_{\hat{D}} \setminus \overline{D}$ yields the Dirichlet-to-Neumann map $\hat{\Lambda}_\kappa$ on $\partial \hat{D}$.

The forward problem of EIT can be modeled by different measurement models. In the so-called *continuum model*, one assumes that it is possible to measure the electric potential u on the whole boundary for a prescribed conormal flux through ∂D

$$(3) \quad \partial_{\kappa\nu} u := \kappa\nu \cdot \nabla u \llcorner \partial D = f,$$

where ν denotes the exterior unit normal vector on ∂D and $f \in L^2_\diamond(\partial D)$ a bounded function modeling the signed density of the outgoing current.

The most accurate forward model for real-life impedance tomography is the *complete electrode model*, cf. [13, 39], where under the assumption that measurements are performed using N electrodes E_l , $l = 1, \dots, N$, on the boundary ∂D , the electric potential u satisfies the Robin boundary condition

$$(4) \quad \kappa\nu \cdot \nabla u \llcorner \partial D + gu \llcorner \partial D = f \quad \text{on } \partial D,$$

for piecewise constant functions $g, f : \partial D \rightarrow \mathbb{R}$ given by

$$(5) \quad g := \frac{1}{z} \sum_{l=1}^N [E_l], \quad f := \frac{1}{z} \sum_{l=1}^N U_l [E_l].$$

Here, $[E_l]$ is the indicator function of the l -th electrode and $U = (U_1, \dots, U_N)^T$ denotes the prescribed voltage pattern. The positive constant $z \in \mathbb{R}_+$ is the so-called *contact impedance* which models electrochemical effects at the electrode-object interface. The electrodes in the complete electrode model $E_l \subset \partial D$, $l = 1, \dots, N$, are modeled by disjoint, simply connected, surface patches, each having a smooth boundary curve. Moreover we always assume that the ground voltage has been chosen such that

$$(6) \quad \sum_{l=1}^N U_l = 0.$$

For given voltage pattern $U \in \mathbb{R}^N$ satisfying (6), the equations (1) and (4) define the electric potential $u \in H^1(D)$ uniquely, cf. [39].

2.2. Preliminaries from symmetric Dirichlet space theory. In his seminal paper [17], Fukushima established a one-to-one correspondence between regular symmetric Dirichlet spaces and symmetric Hunt processes. We assume that the reader is familiar with the basic results from the theory of symmetric Dirichlet spaces, as elaborated for instance in the monograph [19]. Let us consider the following symmetric bilinear forms on $L^2(D)$:

$$(7) \quad \mathcal{E}(v, w) = \int_D \kappa \nabla v(x) \cdot \nabla w(x) dx, \quad v, w \in \mathcal{D}(\mathcal{E}) = H^1(D);$$

$$(8) \quad \mathcal{E}_0(v, w) = \int_D \nabla v(x) \cdot \nabla w(x) \, dx, \quad v, w \in \mathcal{D}(\mathcal{E}_0) = H^1(D).$$

It is well-known that (8) is associated with the reflecting Brownian motion (scaled by a factor 2), while (7) corresponds to a general reflecting diffusion process. First recall that we may associate with the Dirichlet space $(\mathcal{D}(\mathcal{E}), \mathcal{E})$ a non-positive definite self-adjoint operator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ such that for $v \in \mathcal{D}(\mathcal{L})$ we have $\langle -\mathcal{L}v, w \rangle = \mathcal{E}(v, w)$ for all $w \in \mathcal{D}(\mathcal{E})$ and

$$\mathcal{D}(\mathcal{L}) = \left\{ v \in \mathcal{D}(\mathcal{E}) : \exists \phi \in L^2(D) \text{ s.t. } \mathcal{E}(v, w) = \int_D \phi w \, dx \, \forall w \in \mathcal{D}(\mathcal{E}) \right\}.$$

This Dirichlet space is regular on $L^2(D)$, i.e., $\mathcal{D}(\mathcal{E}) \cap C(\overline{D})$ is dense in both, $(C(\overline{D}), \|\cdot\|_\infty)$ and $(\mathcal{D}(\mathcal{E}), \|\cdot\|_{\mathcal{E}_1})$, where $\|\cdot\|_{\mathcal{E}_\beta} := \sqrt{\mathcal{E}_\beta(\cdot, \cdot)}$, $\beta > 0$, with $\mathcal{E}_\beta(\cdot, \cdot) := \mathcal{E}(\cdot, \cdot) + \beta \langle \cdot, \cdot \rangle$. This follows directly from the convergence of $v_\alpha := \alpha G_\alpha v$ in $(\mathcal{D}(\mathcal{E}), \|\cdot\|_{\mathcal{E}_1})$ as $\alpha \rightarrow \infty$, where $\{G_\alpha, \alpha > 0\}$ denotes the strongly continuous resolvent on $L^2(D)$ associated with $(\mathcal{D}(\mathcal{E}), \mathcal{E})$, cf. [19], if one has that $v_\alpha \in C(\overline{D})$ for all $\alpha > 0$. This is, for instance, a consequence of our Theorem 3.1. Moreover the Dirichlet space $(\mathcal{D}(\mathcal{E}), \mathcal{E})$ is local in the sense that $\mathcal{E}(v, w) = 0$, whenever $\text{supp}(v)$ and $\text{supp}(w)$ are disjoint compact sets. The *capacity* of an open subset O of \overline{D} is defined by $\text{Cap}(O) = \inf_{v \in \mathcal{D}(\mathcal{E})} \{\mathcal{E}_1(v, v) : v \geq 1 \text{ a.e. on } O\}$ and that of a general subset is given by $\text{Cap}(B) = \inf\{\text{Cap}(O) : O \text{ is open and } B \subset O\}$. Cap is a Choquet capacity, cf. [19], and a Borel set $B \subset \overline{D}$ is called \mathcal{E} -exceptional if $\text{Cap}(B) = 0$. The general theory of regular local symmetric Dirichlet spaces yields that there exist an \mathcal{E} -exceptional set $\mathcal{N} \subset \overline{D}$ and a conservative diffusion process $\mathbf{X} = (\Omega, \mathcal{F}, \{\mathbf{X}_t, t \geq 0\}, \mathbb{P}_x)$, starting from every $x \in \overline{D} \setminus \mathcal{N}$ (denoted ‘for quasi every (abbreviated q.e.) $x \in \overline{D}$ ’), properly associated with $(\mathcal{D}(\mathcal{E}), \mathcal{E})$. That is, for every non-negative $\phi \in L^2(D)$ the transition semigroup $P_t \phi(x) := \mathbb{E}_x \phi(\mathbf{X}_t)$, $x \in \overline{D} \setminus \mathcal{N}$, of \mathbf{X} is a version of the strongly continuous semigroup $T_t \phi$ of contractions on $L^2(D)$ associated with $(\mathcal{D}(\mathcal{E}), \mathcal{E})$. Note that \mathbf{X} is in general not a semimartingale and that it is not known in general where the \mathcal{E} -exceptional set \mathcal{N} is located. The latter imposes a severe limitation on practical applications which we have to remove here.

For convenience of the reader let us recall the definition of additive functionals of Markov processes depending on the potential theory of the bilinear form \mathcal{E} from [19]. Let $\{\mathcal{F}_t, t \geq 0\}$ denote the minimal augmented filtration generated by \mathbf{X} and without loss of generality let us assume that \mathbf{X} is defined on the canonical sample space $\Omega = C([0, \infty); \overline{D})$ on which the time shift operator Θ is well-defined by $\mathbf{X}_s(\Theta_t(\omega)) = \mathbf{X}_{t+s}(\omega)$, $s, t \geq 0$.

Definition 2.1. A real-valued stochastic process $A = (\Omega, \mathcal{F}, \{A_t, t \geq 0\}, \mathbb{P}_x)$ is an *additive functional* (abbreviated AF) of \mathbf{X} if the following conditions hold:

- (i) A_t is adapted to \mathcal{F}_t for every $t \geq 0$;
- (ii) There exists a defining set $\Lambda \in \mathcal{F}_\infty$ and an \mathcal{E} -exceptional set $\mathcal{N} \subset \overline{D}$ such that $\mathbb{P}_x(\Lambda) = 1$ for $x \in \overline{D} \setminus \mathcal{N}$ and $\Theta_t \Lambda \subset \Lambda$ for every $t \geq 0$;
- (iii) For $\omega \in \Lambda$, $A_t(\omega)$ is right-continuous and has left limits in $t \in [0, \infty)$ with $A_0(\omega) = 0$ and $|A_t(\omega)| < \infty$ for every $t < \infty$;
- (iv) $A_{t+s}(\omega) = A_t(\omega) + A_s(\Theta_t \omega)$.

If in addition the mapping $t \mapsto A_t(\omega)$ is positive and continuous for each $\omega \in \Lambda$, then A is a *positive continuous additive functional* (abbreviated PCAF). An additive functional admitting a defining set Λ with $\mathbb{P}_x(\Lambda) = 1$ for all $x \in \overline{D}$ is called an *additive functional in the strict sense*.

Definition 2.2. A positive Borel measure λ on \overline{D} is called a *smooth measure* of \mathbf{X} if the following conditions hold:

- (i) λ charges no sets of zero capacity, i.e., $\lambda(\mathcal{N}) = 0$ if \mathcal{N} is \mathcal{E} -exceptional;
- (ii) There exists an increasing sequence $\{C_k\}_{k \in \mathbb{N}}$ of closed sets satisfying

$$\lim_{k \rightarrow \infty} \lambda(K \setminus C_k) = 0 \text{ for every compact set } K,$$

such that $\lambda(C_k) < \infty$ and $\lambda(\overline{D} \setminus \cup_{k \in \mathbb{N}} C_k) = 0$.

We can now formulate the well-known *Revuz correspondence*. The family \mathcal{A}_c^+ of all PCAFs of \mathbf{X} and the family \mathcal{S} of all smooth measures on \overline{D} are in one-to-one correspondence. In other words, for every $A \in \mathcal{A}_c^+$ there exists a unique $\lambda \in \mathcal{S}$, and vice versa, that satisfy

$$(9) \quad \lim_{t \rightarrow 0+} \frac{1}{t} \int_D \mathbb{E}_x \left\{ \int_0^t \phi(\mathbf{X}_s) dA_s \right\} \psi(x) dx = \int_{\overline{D}} \phi(x) \psi(x) d\lambda(x),$$

for every non-negative Borel measurable function ϕ and γ -excessive function ψ , $\gamma \geq 0$. Recall that a non-negative function ψ is called γ -excessive (with respect to \mathbf{X}) if $\lim_{t \rightarrow 0+} e^{-\gamma t} \mathbb{E}_x \psi(\mathbf{X}_t) = \psi(x)$ for $x \in \overline{D}$. By the Lipschitz property of ∂D , the Lebesgue surface measure σ on ∂D exists and it is easy to see that σ is a smooth measure of \mathbf{X} . Let thus L denote the PCAF of \mathbf{X} whose Revuz measure is given by σ . In analogy to the notion for case of smooth coefficients, see, e.g., [25, 36, 7], we call L the *boundary local time* of the reflecting diffusion process \mathbf{X} .

Let us conclude this section by recalling that in the framework of symmetric Dirichlet spaces, the celebrated Fukushima decomposition and the corresponding transformation formula, cf. [19], play in some sense the roles of the Doob-Meyer decomposition and Itô's formula for semimartingales: If $v \in \mathcal{D}(\mathcal{E})$, then the composite process $v(\mathbf{X}) = (\Omega, \mathcal{F}, \{v(\mathbf{X}_t), t \geq 0\}, \mathbb{P}_x)$ admits the following unique decomposition

$$(10) \quad \tilde{v}(\mathbf{X}_t) = \tilde{v}(\mathbf{X}_0) + M_t^{[v]} + N_t^{[v]}, \quad \text{for all } t > 0, \quad \mathbb{P}_x\text{-a.s.},$$

holding for q.e. $x \in \overline{D}$, where \tilde{v} is a quasi-continuous version of v , $M^{[v]}$ is a martingale AF of \mathbf{X} having finite energy and $N^{[v]}$ is a continuous AF of \mathbf{X} having zero energy. A function ϕ which is defined q.e. on \overline{D} is called *quasi-continuous* if for every $\varepsilon > 0$ there is an open subset $O \subset \overline{D}$ with $\text{Cap}(O) < \varepsilon$ such that $\phi|_{\overline{D} \setminus O}$ is continuous. It is well-known, that every $v \in \mathcal{D}(\mathcal{E})$ has a quasi-continuous version. Recall moreover that the *energy* of an AF A is defined as

$$\lim_{t \rightarrow 0+} \frac{1}{2t} \int_D \mathbb{E}_x A_t^2 dx$$

and that the elements of the set

$$\{A : A \text{ is an AF of } \mathbf{X} \text{ with } \mathcal{E}\text{-exceptional set } \mathcal{N} \text{ s.t.}$$

$$\mathbb{E}_x A_t^2 < \infty \text{ for all } t > 0 \text{ and } \mathbb{E}_x A_t = 0 \text{ for all } x \in \overline{D} \setminus \mathcal{N}\}$$

are called *martingale AFs*.

3. REFINEMENT OF THE REFLECTING DIFFUSION PROCESS

In order to refine the diffusion process \mathbf{X} to start from every $x \in \overline{D}$ without exceptional set we need the well-known connection between the strongly continuous sub-Markovian semigroup $\{T_t, t \geq 0\}$ on $L^2(D)$ and the evolution system corresponding to

$(\mathcal{L}, \mathcal{D}(\mathcal{L}))$, see, e.g., the monograph [37]. Namely for every $v_0 \in L^2(D)$, the function $v(t) := T_t v_0$ belongs to the function space $W(0, T; H^1(D), H^{-1}(D))$ given by the set

$$\{\phi \in L^2((0, T); H^1(D)) : \dot{\phi} \in L^2((0, t); H^{-1}(D))\}$$

and is the unique solution of the abstract Cauchy problem

$$(11) \quad \begin{aligned} \dot{v} + \mathcal{L}v &= 0 \quad \text{in } (0, T) \\ v(0) &= v_0. \end{aligned}$$

On the other hand, given (11) it is not difficult to verify that v also satisfies the parabolic equation

$$(12) \quad - \int_0^T \langle v(t), w \rangle \dot{\varphi}(t) dt + \int_0^T \langle \mathcal{L}v(t), w \rangle \varphi(t) dt - \langle v_0, w \rangle \varphi(0) = 0$$

for all $w \in H^1(D)$ and all $\varphi \in C_c^\infty([0, T])$. Moreover it is well-known that T_t is a bounded operator from $L^1(D)$ to $L^\infty(D)$ for every $t > 0$. Therefore by the Dunford-Pettis Theorem, it can be represented as an integral operator for every $t > 0$,

$$(13) \quad T_t \phi(x) = \int_D p(t, x, y) \phi(y) dy \quad \text{a.e. on } D, \text{ for every } \phi \in L^1(D),$$

where for all $t > 0$ we have $p(t, x, y) \in L^\infty(D \times D)$ and $p(t, x, y) \geq 0$ a.e..

The following Theorem generalizes a well-known result for diffusion processes on \mathbb{R}^d from [40].

Theorem 3.1. *For each fixed $0 < t_0 \leq T$ there exist positive constants c_1 and c_2 such that*

$$(14) \quad |p(t_2, x_2, y_2) - p(t_1, x_1, y_1)| \leq c_1(\sqrt{t_2 - t_1} + |x_2 - x_1| + |y_2 - y_1|)^{c_2}$$

for all $t_0 \leq t_1 \leq t_2 \leq T$ and all $(x_1, y_1), (x_2, y_2) \in \overline{D} \times \overline{D}$.

Proof. The main idea of the proof is the following extension by reflection technique, see for example [42, Section 2.4.3]: We extend the solution of a parabolic problem by reflection at the boundary, then show that this extension again solves a parabolic problem so that we can apply a well-known interior regularity result.

First note that Nash's inequality holds for the underlying Dirichlet space $(\mathcal{D}(\mathcal{E}), \mathcal{E})$, i.e., there exists a constant $c_1 > 0$ such that

$$\|v\|^{2+4/d} \leq c_1(\mathcal{E}(v, v) + \|v\|^2) \|v\|_{L^1(D)}^{4/d} \quad \text{for all } v \in H^1(D).$$

This is a direct consequence of the uniform ellipticity (2) and [4, Corollary 2.2], where Nash's inequality is shown to hold for the Dirichlet space $(H^1(D), \mathcal{E}_0)$ with D a bounded Lipschitz domain. Analogously to the proof of [4, Theorem 3.1], it follows thus from [8, Theorem 3.25] that the transition kernel density satisfies an Aronson type Gaussian upper bound

$$(15) \quad p(t, x, y) \leq c_1 t^{-d/2} \exp\left(-\frac{|x - y|^2}{c_2 t}\right) \quad \text{for all } t \leq 1 \text{ and } (x, y) \in \overline{D} \times \overline{D}.$$

In particular $\sup_{0 < t \leq 1} \|p(t, \cdot, \cdot)\|_\infty$ is finite and by Nash's interior Hölder continuity Theorem, cf. [34], the estimate (14) is true for every $(x_1, y_1), (x_2, y_2)$ with $d(x_i, \partial D), d(y_i, \partial D) > c_3$, $i = 1, 2$, for some constant $c_3 > 0$ and all $t_0 \leq t_1 \leq t_2 \leq 1$. Note that

by the Markov property of the semigroup the Chapman-Kolmogorov equation holds, i.e.,

$$p(t_1 + t_2, x, y) = \int_D p(t_1, x, z) p(t_2, z, y) \, dz$$

for every t_1, t_2 and a.e. $x, y \in \overline{D}$, in particular for fixed $y \in \overline{D}$ the function $v := p(\cdot, \cdot, y)$ is the unique solution to (11) with initial value $v_0 := p(0, \cdot, y) \in L^2(D)$. Now let $z \in \partial D$ so that by the Lipschitz property of ∂D we have after translation and rotation $B(z, r_D) \cap \overline{D} = \{(\tilde{x}, x_d) \in B(z, r_D) : x_d > \gamma(\tilde{x})\}$ and $B(z, r_D) \cap \partial D = \{\tilde{x} \in B(z, r_D) : x_d = \gamma(\tilde{x})\}$, where we have introduced the notation $\tilde{x} = (x_1, \dots, x_{d-1})^T$. Let us furthermore introduce the one-to-one transformation $\Psi(x) := (\tilde{x}, x_d - \gamma(\tilde{x}))$ which maps $B(z, r_D) \cap \overline{D}$ into a subset of the hyperplane $\{(\tilde{y}, 0)\}$ and straightens the boundary $B(z, r_D) \cap \partial D$. Ψ is a bi-Lipschitz transformation and the Jacobians of both Ψ and Ψ^{-1} are bounded with bounds that depend only on the Lipschitz constant c_D . Since v is the solution of (11) with appropriate initial condition, the function $\hat{v} := v(\cdot, \Psi^{-1}(\cdot))$ must satisfy the following parabolic equation in $\hat{D}(z, r_D) := \Psi(B(z, r_D) \cap \overline{D})$:

$$\begin{aligned} \int_0^T \dot{\varphi}(t) \int_{\hat{D}(z, r_D)} \hat{v}(t) w \, dx \, dt &= - \sum_{i,j=1}^d \int_0^T \varphi(t) \int_{\hat{D}(z, r_D)} \hat{\kappa}_{ij} \partial_i \hat{v}(t) \partial_j w \, dx \, dt \\ &\quad - \varphi(0) \int_{\hat{D}(z, r_D)} \hat{v}_0 w \, dx \end{aligned}$$

for all $w \in C_c^\infty(\hat{D}(z, r_D))$ and all $\varphi \in C_c^\infty([0, T])$. The coefficient $\hat{\kappa}$ is obtained via change of variables and it is bounded and uniformly elliptic by the boundedness of the Jacobians of Ψ and Ψ^{-1} , respectively. Now we use reflection on the hyperplane $\{(\tilde{y}, 0)\}$ via the mapping $\rho(x) := (\tilde{x}, -x_d)$ which yields that the function $\hat{v}(\cdot, \rho(\cdot))$ satisfies a parabolic equation on $\rho(\hat{D}(z, r_D))$. Summing up both parabolic equations on $\hat{D}(z, r_D)$ and on $\rho(\hat{D}(z, r_D))$, respectively, we obtain that the function

$$\check{v}(t, x) := \begin{cases} \hat{v}(t, x), & x \in \hat{D}(z, r_D) \\ \hat{v}(t, \rho(x)), & x \in \rho(\hat{D}(z, r_D)) \end{cases}$$

satisfies a parabolic equation in $(\hat{D}(z, r_D) \cup \rho(\hat{D}(z, r_D)))$. By Nash's interior Hölder estimate for \check{v} , together with the fact that we may choose $c_3 = \frac{r_D}{4c_D}$, we obtain thus

$$|p(t_2, x_2, \Psi^{-1}(y_2)) - p(t_1, x_1, \Psi^{-1}(y_1))| \leq c_1(\sqrt{t_2 - t_1} + |y_2 - y_1|)^{c_2}$$

for all $t_0 \leq t_1 \leq t_2 \leq 1$ and $y_1, y_2 \in \{(\tilde{x}, x_d) : |\tilde{x}| < c_3, x_d \in (0, r_D/4)\}$. As Ψ is bi-Lipschitz, for fixed x , the mapping $(t, y) \mapsto p(t, x, y)$ is Hölder continuous in $(t_0, 1] \times (B(z, c_3) \cap \overline{D})$ and by symmetry of the transition kernel density the same holds true for the mapping $(t, x) \mapsto p(t, x, y)$ for fixed y . Finally, the assertion on $(t_0, 1] \times \overline{D} \times \overline{D}$ follows due to compactness of ∂D and its generalization to arbitrary $T > 0$ is obtained after repeatedly applying Chapman-Kolmogorov. \square

By [18, Theorem 2] the Hölder continuity of the transition density kernel ensures that we may refine the process \mathbf{X} to start from every $x \in \overline{D}$ by identifying the strongly continuous semigroup $\{T_t, t \geq 0\}$ with the transition semigroup $\{P_t, t \geq 0\}$. Moreover the decomposition (10) holds \mathbb{P}_x -a.s. for every $x \in \overline{D}$ if v is continuous and locally in

$H^1(D)$ and the energy measure of $M^{[v]}$ given by

$$d\mu_{\langle v \rangle}(x) := 2 \sum_{i,j=1}^d \kappa_{ij}(x) \partial_i v(x) \partial_j v(x) dx$$

is a smooth measure of \mathbf{X} in the strict sense, that is, there is an increasing sequence of finely open sets $\{D_k, k \geq 1\}$ so that $\cup_{k \in \mathbb{N}} D_k = \overline{D}$, $[D_k] \mu_{\langle v \rangle}$ is a finite Borel measure and the 1-resolvent $G_1([D_k] \mu_{\langle v \rangle})$ is bounded for every $k \geq 1$. In this case, both $M^{[v]}$ and $N^{[v]}$ can be taken to be additive functionals of \mathbf{X} in the strict sense.

4. FEYNMAN-KAC TYPE REPRESENTATION FORMULAE

Let us first state some auxiliary Lemmata.

Lemma 4.1. *The transition density kernel p approaches the stationary distribution uniformly and exponentially fast, that is, there exists a $t_0 > 0$ and a constant $c_3 > 0$ such that for all $(x, y) \in \overline{D} \times \overline{D}$ and every $t \geq t_0$,*

$$(16) \quad |p(t, x, y) - |D|^{-1}| \leq \exp(-c_3 t).$$

Proof. It is well-known that there exists an orthonormal basis $\{\phi_j : j \in \mathbb{N}\}$ of $L^2(D)$ and an increasing sequence of constants $(\lambda_j)_{j \in \mathbb{N}}$ such that $\lambda_0 = 0$, $\lambda_1 > 0$ and the functions ϕ_j are the weak solutions of the eigenvalue problem

$$\begin{cases} -\nabla \cdot (\kappa \nabla \phi_j) = \lambda_j \phi_j & \text{in } D, \\ \kappa \nu \cdot \nabla \phi_j = 0 & \text{on } \partial D. \end{cases}$$

Using Theorem 3.1 it is not difficult to see that the eigenfunctions satisfy $\phi_j \in C(\overline{D})$ and for every $t > 0$ and $x, y \in \overline{D}$ we have

$$p(t, x, y) = |D|^{-1} + \sum_{j=1}^{\infty} e^{-\lambda_j t} \phi_j(x) \phi_j(y)$$

by the fact that for every $v_0 \in L^2(D)$, the function $v(t)$ given by

$$v(t) = \sum_{j=0}^{\infty} e^{-\lambda_j t} \langle v_0, \phi_j \rangle \phi_j$$

is the solution of the abstract Cauchy problem (11). Using this eigenexpansion, one can deduce the assertion in a straightforward manner from the Aronson type Gaussian bound (15), cf. [4] for a proof when \mathbf{X} is the reflecting Brownian motion. \square

Lemma 4.2. *The set*

$$(17) \quad V(D) := \{\phi : \phi \in C^2(D), \partial_\nu \phi = 0 \text{ a.e. on } \partial D\}$$

is dense in $H^1(D)$.

Proof. Diagonalizing the Neumann Laplacian on D we obtain an orthonormal basis $\{\phi_j, j \in \mathbb{N}\}$ of $L^2(D)$ and an increasing sequence $(\lambda_j)_{j \in \mathbb{N}}$ of real positive numbers which tend to infinity such that for every $j \in \mathbb{N}$, $\phi_j \in H^1(D)$ is a weak solution of the eigenvalue problem for the Neumann Laplacian. Let now $\psi \in H^1(D)$ such that $\langle \psi, \phi_j \rangle_{H^1(D)} = 0$ for every $j \in \mathbb{N}$, then

$$\int_D \nabla \phi_j \cdot \nabla \psi dx + \int_D \phi_j \psi dx = \lambda_j \int_D \phi_j \psi dx + \int_D \phi_j \psi dx = 0, \quad \text{for all } j \in \mathbb{N}.$$

The fact that $\{\phi_j, j \in \mathbb{N}\}$ is an orthonormal basis of $L^2(D)$ implies $\psi \equiv 0$ which proves the assertion. \square

Lemma 4.3. *The boundary local time L of \mathbf{X} corresponding to the surface measure σ exists as PCAF in the strict sense.*

Proof. We have to show that the surface measure σ of ∂D has a bounded 1-potential, then the assertion follows immediately from [19, Theorem 5.1.6]. Since we have a continuous transition density kernel p for every $x \in \overline{D}$ and $t > 0$, we know that the 1-potential coincides with the 1-resolvent $G_1\sigma$ of the measure σ given by

$$G_1\sigma(x) = \int_{\partial D} \int_0^\infty p(t, x, y) e^{-t} dt d\sigma(y).$$

We need to show that this is uniformly bounded. According to Lemma 4.1, the transition density kernel p is uniformly bounded for every $t \geq t_0$. This together with Hölder continuity implies

$$c_6 := \sup_{x \in \overline{D}} \int_{\partial D} \int_1^\infty p(t, x, y) e^{-t} dt d\sigma(y) < \infty.$$

Therefore, it is enough to show that

$$\sup_{x \in \overline{D}} \int_{\partial D} \int_0^1 p(t, x, y) dt d\sigma(y) < \infty.$$

For every $\rho > 0$ and every $x \in D$ such that $d(x, \partial D) \geq \rho$ the Gaussian upper bound for the density p gives

$$G_1\sigma(x) \leq c_6 + c_3 \int_0^1 t^{-d/2} e^{-\rho^2/2t} dt.$$

It is straightforward to show that the integrand on the right-hand side has an upper bound $c_4([d \geq 3]\rho^{2-d} + [d = 2]\log \rho^{-1})$. Hence for every fixed $\rho > 0$ we have a uniform upper bound $c_7(\rho)$ for the 1-resolvent.

Since the boundary ∂D is compact we can find $\rho > 0$ and a finite number of balls $B(x_k, 2\rho)$ with centers $x_k \in \partial D$ that cover the set $\{x \in \overline{D} : d(x, \partial D) < \rho\}$. Moreover the ρ can be taken so small that there exist bi-Lipschitz homeomorphisms between these balls and the subsets of \mathbb{R}^d straightening the boundary as in the proof of Theorem 3.1.

By the first part of the proof, we may assume $x \in B(x_k, 2\rho)$ for some k . For every l such that $x \notin B(x_l, 2\rho)$ the previous estimate holds and we have

$$\sup_{x \notin B(x_l, 2\rho)} \int_0^1 \int_{\partial D} [y \in B(x_l, 2\rho)] p(t, x, y) d\sigma(y) dt \leq c_7(\rho).$$

Hence it is enough to show that the contribution coming from the integration over those balls $B(x_l, 2\rho)$ that contain x is finite.

When $x \in B_l := B(x_l, 2\rho)$ we use the Gaussian upper bound for a fixed $t \leq 1$ and Lipschitz change of coordinates estimate $\sigma(\Psi E) \leq c_D^{d-1} \sigma(E)$ of the $(d-1)$ -dimensional surface measure σ . Therefore,

$$\int_{\partial D} [y \in B_l] e^{-|x-y|/2t} d\sigma(y) \leq c_D^{d-1} \int_{\mathbb{R}^{d-1}} [y \in \tilde{B}_l] e^{-|x^*-y^*|/2t} dy$$

where $\tilde{B}_l := B(x_l^*, 4c_D\rho) \subset \mathbb{R}^{d-1}$ is a $(d-1)$ -dimensional ball of radius $4c_D\rho$ and x^* (similarly x_l^* and y^*) is the point x in the new coordinate system. The integrand is

maximized if we move x^* into the center x_l^* of \tilde{B}_l . Therefore, we can estimate

$$\sup_{x \in B(x_l, 2\rho)} \int_{\partial D} [y \in B_l] t^{-d/2} e^{-|x-y|^2/2t} d\sigma(y) \leq c_8 \int_0^\infty t^{-d/2} r^{d-2} e^{-r^2/2t} dr.$$

Again this is straightforward to estimate and we see that integrand on the right-hand side has an upper bound $c_5 t^{-1/2}$. Since $t^{-1/2}$ is integrable at zero, the claim follows. \square

Lemma 4.4. *For every $x \in \overline{D}$ and every bounded Borel measurable function ϕ on ∂D the following occupation formula holds:*

$$(18) \quad \mathbb{E}_x \int_0^t \phi(\mathbf{X}_s) dL_s = \int_0^t \int_{\partial D} p(s, x, y) \phi(y) d\sigma(y) ds \quad \text{for all } t \geq 0.$$

Proof. By Lemma 4.3 the boundary local time of \mathbf{X} exists as a PCAF in the strict sense. Without loss of generality we may assume that ϕ is non-negative. It follows from [19, Theorem 5.1.3] that the Revuz correspondence (9) is equivalent to

$$\mathbb{E}_{\psi \cdot dx} (\phi \cdot L)_t = \int_0^t \langle \phi \cdot \sigma, T_s \psi \rangle ds$$

for every $t > 0$ and all non-negative Borel measurable functions ψ and ϕ . That is,

$$\begin{aligned} \int_D \psi(x) \mathbb{E}_x \int_0^t \phi(\mathbf{X}_s) dL_s dx &= \int_0^t \int_{\partial D} \phi(y) T_s \psi(y) d\sigma(y) ds \\ &= \int_D \psi(x) \int_0^t \int_{\partial D} \phi(y) p(s, x, y) d\sigma(y) ds dx, \end{aligned}$$

where we have used Fubini's Theorem. As this holds for every non-negative Borel measurable function ψ , we may deduce

$$\mathbb{E}_x \int_0^t \phi(\mathbf{X}_s) dL_s = \int_0^t \int_{\partial D} \phi(y) p(s, x, y) d\sigma(y) ds \quad \text{a.e. in } \overline{D}.$$

To obtain the assertion everywhere in \overline{D} consider for $t_0 > 0$ the integral

$$\int_D p(t_0, x, y) \mathbb{E}_y \int_0^T \phi(\mathbf{X}_s) dL_s dy,$$

where we have set $T := t - t_0$. Note that by the Markov property of \mathbf{X} we may write this integral equivalently as

$$\mathbb{E}_x \int_{t_0}^t \phi(\mathbf{X}_s) dL_s = \int_{t_0}^t \int_{\partial D} \phi(z) p(s, x, z) d\sigma(z) ds \quad \text{for every } x \in \overline{D}.$$

Now letting $t_0 \rightarrow 0$ and using the Dominated Convergence Theorem yields the assertion. \square

4.1. Continuum model. The main result for the continuum model (1), (3) is the following Theorem.

Theorem 4.5. *Let κ satisfy (A1) and let $f \in L_\diamond^2(\partial D)$ be bounded. Then there is a unique weak solution $u \in C(\overline{D}) \cap H_\diamond^1(D)$ to the boundary value problem (1), (3). This solution admits the Feynman-Kac type representation*

$$(19) \quad u(x) = \lim_{t \rightarrow \infty} \mathbb{E}_x \int_0^t f(\mathbf{X}_s) dL_s \quad \text{for all } x \in \overline{D}.$$

Proof. The existence of a unique normalized weak solution to (1), (3) is guaranteed by the standard theory of linear elliptic boundary value problems. Let us set $u_t(x) := \mathbb{E}_x \int_0^t f(\mathbf{X}_s) dL_s$ and $u_\infty(x) := \lim_{t \rightarrow \infty} u_t(x)$, $x \in \overline{D}$, respectively. From the occupation formula (18) and the compatibility condition $\int_{\partial D} f(x) d\sigma(x) = 0$ it follows immediately that

$$u_t(x) = \int_0^t \int_{\partial D} (p(t, x, y) - |D|^{-1}) f(y) d\sigma(y) ds \quad \text{for all } x \in \overline{D}.$$

By Lemma 4.1 the convergence towards the stationary distribution is uniform over \overline{D} , in particular,

$$(20) \quad u_\infty(x) = \int_0^\infty \int_{\partial D} (p(t, x, y) - |D|^{-1}) f(y) d\sigma(y) ds \quad \text{for all } x \in \overline{D}.$$

It follows from (20) together with Theorem 3.1 and the Aronson type upper bound (15) that u_∞ is in $C(\overline{D})$. Moreover Lemma 4.1 implies the normalization $\int_D u_\infty(x) dx = 0$.

Now let us use the following regularization technique: Let $(\kappa^{(n)})_{n \in \mathbb{N}}$ denote a sequence of smooth conductivities with components in $C^\infty(\overline{D})$ such that for $1 \leq i, j \leq d$, $\kappa_{ij}^{(n)} \rightarrow \kappa_{ij}$ a.e. as $n \rightarrow \infty$. Let us consider the Dirichlet space $(H^1(D), \mathcal{E}^{(n)})$ with $\mathcal{E}^{(n)}(v, w) := \int_D \kappa^{(n)} \nabla v \cdot \nabla w dx$ and the associated reflecting diffusion process $\mathbf{X}^{(n)}$. Using the Fukushima decomposition (10) for the coordinate functions we obtain the Skorohod decomposition

$$\mathbf{X}_t^{(n)} = x + \int_0^t a^{(n)}(\mathbf{X}_s^{(n)}) ds + \int_0^t B^{(n)}(\mathbf{X}_s) dW_s - \int_0^t \nu(\mathbf{X}_s^{(n)}) dL_s^{(n)},$$

where W is a standard d -dimensional Brownian motion, $a_i^{(n)} := \sum_{j=1}^d \partial_j \kappa_{ij}^{(n)}$, $i = 1, \dots, d$, and the matrix $B^{(n)}$ satisfies $\kappa^{(n)} = \frac{1}{2} B^{(n)} (B^{(n)})^T$. Let us define $u_t^{(n)}$ in the same manner as u_t and $u^{(n)}(x) := \lim_{t \rightarrow \infty} u_t^{(n)}(x)$, $x \in \overline{D}$. We show that $u^{(n)}$ is the unique weak solution of the elliptic boundary value problem $\nabla \cdot (\kappa^{(n)} \nabla u^{(n)}) = 0$ in D with Neumann boundary condition $\partial_\nu u^{(n)} = f$ on ∂D in the Sobolev space $H_\diamond^1(D)$. For test functions $v \in V(D)$ we may apply Itô's formula for semimartingales to obtain

$$\mathbb{E}_x v(\mathbf{X}_t^{(n)}) = v(x) + \mathbb{E}_x \int_0^t \nabla \cdot (\kappa^{(n)} \nabla v(\mathbf{X}_s^{(n)})) ds.$$

By Fubini's Theorem this is equivalent to

$$T_t^{(n)} v(x) - v(x) = \int_0^t \int_D p^{(n)}(s, x, y) \nabla \cdot (\kappa^{(n)} \nabla v(y)) dy ds,$$

where we have used the superscript ' (n) ' for the semigroup and transition density kernel, respectively, corresponding to $\kappa^{(n)}$. Multiplication with f , integration over ∂D and another change of the orders of integration yields finally

$$\int_{\partial D} f(y) (T_t^{(n)} v(y) - v(y)) d\sigma(y) = \left\langle u_t^{(n)}, \nabla \cdot (\kappa^{(n)} \nabla v) \right\rangle.$$

Since $u_t^{(n)} \rightarrow u^{(n)}$ and $T_t^{(n)} v \rightarrow |D|^{-1} \int_D v dx$, both uniformly on \overline{D} , as $t \rightarrow \infty$, we have

$$\left\langle u^{(n)}, \nabla \cdot (\kappa^{(n)} \nabla v) \right\rangle = - \int_{\partial D} f(y) v(y) d\sigma(y),$$

where we have used the expression (20) with $p^{(n)}$ instead of p for $u^{(n)}$. As this holds true for every $v \in V(D)$, $u^{(n)}$ must be the unique normalized weak solution to the boundary value problem by a density argument.

Now let us show the convergence of the sequence $(u^{(n)})_{n \in \mathbb{N}}$ towards $u \in H_\diamond^1(D)$, the unique solution of (1), (3). By the standard Trace Theorem there exists a function $\phi \in H_{\text{div}}^1(D)$ such that $\partial_\nu \phi = f$ and $\mathcal{L}\phi \in (H^1(D))'$. The bilinear form \mathcal{E} is coercive on H_\diamond^1 , thus by the Lax-Milgram Theorem there exists a unique $w \in H_\diamond^1(D)$ satisfying

$$\int_D \kappa \nabla w \cdot \nabla v \, dx = \langle \mathcal{L}\phi, v \rangle_{(H_\diamond^1(D))', H_\diamond^1(D)} \quad \text{for all } v \in H_\diamond^1(D),$$

i.e., w is the weak solution of the problem $\mathcal{L}w = -\mathcal{L}\phi$ with homogeneous Neumann boundary condition and thus u has the form $u = w + \phi$. Analogously, $u^{(n)}$ has the form $u^{(n)} = w^{(n)} + \phi$. We show that $\mathcal{L}^{(n)}\phi \rightarrow \mathcal{L}\phi$ in the norm of $(H_\diamond^1(D))'$. We have for every $v \in H_\diamond^1(D)$

$$\langle \mathcal{L}\phi - \mathcal{L}^{(n)}\phi, v \rangle_{(H_\diamond^1(D))', H_\diamond^1(D)} = \sum_{i,j=1}^d \int_D (\kappa_{ij}^{(n)} - \kappa_{ij}) \partial_j \phi \partial_i v \, dx.$$

Notice that $(\kappa_{ij}^{(n)} - \kappa_{ij}) \partial_j \phi \in L^2(D)$, $1 \leq i, j \leq d$, hence the Dominated Convergence Theorem yields

$$\|(\kappa_{ij}^{(n)} - \kappa_{ij}) \partial_j \phi\|_{L^2(D)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

After applying Hölder's inequality we have thus shown that

$$(21) \quad \|\mathcal{L}\phi - \mathcal{L}^{(n)}\phi\|_{(H_\diamond^1(D))'} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Moreover from our assumptions on the sequence $(\kappa^{(n)})_{n \in \mathbb{N}}$ it is clear that for $1 \leq i, j \leq d$, the functions $|\kappa_{ij}^{(n)} - \kappa_{ij}|^2$ are measurable and bounded and $|\kappa_{ij}^{(n)} - \kappa_{ij}|^2 \rightarrow 0$ for a.e. $x \in \overline{D}$ as $n \rightarrow \infty$. Hence the Dominated Convergence Theorem yields $\|\kappa_{ij}^{(n)} - \kappa_{ij}\|_{L^2(D)} \rightarrow 0$ as $n \rightarrow \infty$. It is well-known, that this implies G -convergence of the sequence of elliptic operators $(\mathcal{L}^{(n)})_{n \in \mathbb{N}}$ towards \mathcal{L} , cf. [45]. By [45, Theorem 5] this G -convergence together with the convergence (21) yields that $w^{(n)} \rightarrow w$ weakly in $H_\diamond^1(D)$, thus implying $u^{(n)} \rightarrow u$ weakly in $H_\diamond^1(D)$.

On the other hand by [38, Lemma 2.2] together with the Hölder continuity up to the boundary of both, $p^{(n)}$, $n \in \mathbb{N}$, and p , it follows that for fixed $x \in \overline{D}$, $p^{(n)}(\cdot, x, \cdot) \rightarrow p(\cdot, x, \cdot)$ uniformly on compacts in $(0, T] \times \overline{D}$ for all $T > 0$. It follows from (20) that $u^{(n)}(x) \rightarrow u_\infty(x)$ for all $x \in \overline{D}$ as $n \rightarrow \infty$. In particular u must coincide with u_∞ and the assertion is proved. \square

Remark 2. Note that a similar regularization technique may be used to prove the Feynman-Kac formula

$$u(x) = \mathbb{E}_x \phi(\mathbf{X}_{\tau_D}), \quad x \in \overline{D}$$

for the conductivity equation (1) with Dirichlet boundary condition $u|_{\partial D} = \phi$, where τ_D denotes the first exit time from the domain D . Such a proof requires the fact that for every $x \in \overline{D}$,

$$\mathbb{P}_x \circ (\mathbf{X}^{(n)}, \tau_D^{(n)})^{-1} \rightarrow \mathbb{P}_x \circ (\mathbf{X}, \tau_D)^{-1} \quad \text{as } n \rightarrow \infty$$

in the topology of $C([0, \infty); \mathbb{R}^d) \times \mathbb{R}$ which follows easily from the assumption that D has a Lipschitz boundary, i.e., all points of ∂D are regular, cf. [28, Section 4.27].

4.2. Complete electrode model. The main result for the complete electrode model (1), (4) is the following Theorem.

Theorem 4.6. *For given functions f, g defined by (5) and a voltage pattern $U \in \mathbb{R}^N$ satisfying (6), there is a unique weak solution $u \in C(\overline{D}) \cap H^1(D)$ to the boundary value problem (1), (4). This solution admits the Feynman-Kac type representation*

$$(22) \quad u(x) = \mathbb{E}_x \int_0^\infty e_g(t) f(\mathbf{X}_t) dL_t \quad \text{for all } x \in \overline{D},$$

with

$$(23) \quad e_g(t) := \exp \left(- \int_0^t g(\mathbf{X}_s) dL_s \right), \quad t \geq 0.$$

Before we are ready to give a proof of Theorem 4.6 let us introduce the *Feynman-Kac semigroup* of the complete electrode model, i.e., the one-parameter family of operators $\{T_t^g, t \geq 0\}$ defined by

$$(24) \quad T_t^g v(x) := \mathbb{E}_x e_g(t) v(\mathbf{X}_t), \quad x \in \overline{D} \text{ and } t \geq 0.$$

The following Theorem is crucial for proving the claimed regularity of the potential.

Theorem 4.7. *$\{T_t^g, t \geq 0\}$ is a strong Feller semigroup on $L^2(D)$.*

Proof. To show that $\{T_t^g, t \geq 0\}$ is a strongly continuous semigroup on $L^2(D)$, one can employ the theory of symmetric Dirichlet spaces. To be precise, one must show that $\{T_t^g, t \geq 0\}$ is associated with the perturbed Dirichlet space $(\mathcal{D}(\mathcal{E}^g), \mathcal{E}^g)$, which is obtained by perturbation of $(\mathcal{D}(\mathcal{E}), \mathcal{E})$ with the measure $-g \cdot \sigma$, i.e.,

$$\mathcal{E}^g(v, w) = \mathcal{E}(v, w) + \int_{\partial D} g(x) v(x) w(x) d\sigma(x), \quad v, w \in \mathcal{D}(\mathcal{E}^g) = H^1(D),$$

where the identity $\mathcal{D}(\mathcal{E}^g) = H^1(D)$ follows from the standard Trace Theorem. As in the proof of [19, Theorem 6.1.1], it is sufficient to show $G_\alpha^g \phi \in H^1(D)$, $\mathcal{E}_\alpha^g(G_\alpha^g \phi, v) = \langle \phi, v \rangle$ for all $\phi \in L^2(D)$ and $v \in H^1(D)$, where $G_\alpha^g \phi$ denotes the Laplace transform

$$G_\alpha^g \phi(x) = \mathbb{E}_x \int_0^\infty e_g(t) e^{-\alpha t} \phi(\mathbf{X}_t) dt.$$

We omit this computation for brevity. Moreover T_t^g is a bounded operator from $L^1(D)$ to $L^\infty(D)$ for every $t > 0$ which can be shown using Fatou's Lemma. By the Dunford-Pettis Theorem T^g can thus be represented as an integral operator for every $t > 0$,

$$(25) \quad T_t^g \phi(x) = \int_D p^g(t, x, y) \phi(y) dy \quad \text{a.e. on } D, \text{ for every } \phi \in L^1(D),$$

where for all $t > 0$ we have $p^g(t, x, y) \in L^\infty(D \times D)$ and $p^g(t, x, y) \geq 0$ for a.e. $x, y \in \overline{D}$. For the strong Feller property we have to show that $T_t^g, t > 0$ maps bounded measurable functions to $C(\overline{D})$. We use the method from the papers [25, 36] to construct the transition kernel density p_g . Let $p_0^g(t, x, y) := p(t, x, y)$ and set

$$p_k^g(t, x, y) := \int_0^t \int_{\partial D} p(s, x, z) g(z) p_{k-1}^g(t-s, z, y) d\sigma(z) ds, \quad k \in \mathbb{N}.$$

Note that the terms p_k^g are positive and symmetric in the x and y variables by the properties of p . By induction using Lemma 4.4 it is not difficult to verify that for all $k \in \mathbb{N}$

$$\int_0^t \int_{\partial D} g(x) p_k^g(s, x, y) d\sigma(x) ds \leq \left(\sup_{x \in \overline{D}} \left\{ \mathbb{E}_x \int_0^t g(\mathbf{X}_s) dL_s \right\} \right)^{k+1}$$

and that there is a positive constant c_1 such that

$$(26) \quad p_k^g(t, x, y) \leq c_1^{k+1} t^{-d/2} \left(\sup_{x \in \overline{D}} \left\{ \mathbb{E}_x \int_0^t g(\mathbf{X}_s) dL_s \right\} \right)^{k+1} \quad \text{for all } k \in \mathbb{N}.$$

Let us show the continuity of p_k^g , $k \in \mathbb{N} \cup \{0\}$. For $k = 0$ this is a consequence of Theorem 3.1. Now assume that p_{k-1}^g is continuous on $(t_0, T] \times \overline{D} \times \overline{D}$ for $t_0 > 0$, then we have for $t \in (t_0, T]$

$$\begin{aligned} p_k^g(t, x, y) &= \int_0^{t_0} \int_{\partial D} p(s, x, z) g(z) p_{k-1}^g(t-s, z, y) d\sigma(z) ds \\ &\quad + \int_{t_0}^t \int_{\partial D} p(s, x, z) g(z) p_{k-1}^g(t-s, z, y) d\sigma(z) ds. \end{aligned}$$

Note that the first integral on the right-hand side tends to zero uniformly as $t_0 \rightarrow 0$, which is a consequence of (26), while the second integral is continuous by assumption. Hence there exists a $T > 0$ such that the series $p^g(t, x, y) := \sum_{k=0}^{\infty} p_k^g(t, x, y)$ converges absolutely and uniformly on any compact subset of $(0, T] \times \overline{D} \times \overline{D}$ and is thus continuous on $(0, T] \times \overline{D} \times \overline{D}$. By the Markov property we have for all $t \in (0, T]$ and every $x \in \overline{D}$ the following expression for $T_t^g \phi(x)$:

$$\int_D p^g(t, x, y) \phi(y) dy = \mathbb{E}_x \phi(\mathbf{X}_t) + \sum_{k=1}^{\infty} \frac{1}{k!} \mathbb{E}_x \left\{ \left(\int_0^t g(\mathbf{X}_s) dL_s \right)^k \phi(\mathbf{X}_t) \right\}.$$

The assertion for arbitrary $T > 0$ follows from the Chapman-Kolmogorov equation. \square

Proof of Theorem 4.6. First we show that $u \in C(\overline{D})$. Let us define a martingale with respect to $\{\mathcal{F}_t, t \geq 0\}$ by

$$\mathbb{E}_x \left\{ \int_0^{\infty} e_g(s) f(\mathbf{X}_s) dL_s F_t \right\} = \int_0^t e_g(s) f(\mathbf{X}_s) dL_s + e_g(t) u(\mathbf{X}_t),$$

where the right-hand side is obtained using the Markov property of \mathbf{X} together with the fact that e_g is a multiplicative functional of \mathbf{X} . Obviously

$$e_g(t) u(\mathbf{X}_t) - u(x) + \int_0^t e_g(s) f(\mathbf{X}_s) dL_s$$

is a martingale with respect to $\{\mathcal{F}_t, t \geq 0\}$ as well and hence we have for all $0 \leq s \leq t$:

$$e_g(s) u(\mathbf{X}_s) = e_g(s) \mathbb{E}_{\mathbf{X}_s} e_g(t-s) u(\mathbf{X}_{t-s}) + e_g(s) \mathbb{E}_{\mathbf{X}_s} \int_0^{t-s} e_g(r) f(\mathbf{X}_r) dL_r.$$

Setting $s = 0$ yields thus

$$u(x) = T_t^g u(x) + \mathbb{E}_x \int_0^t e_g(s) f(\mathbf{X}_s) dL_s \quad \text{for all } t \geq 0.$$

By the Markov property we have $T_t^g u(x) = T_s^g(T_{t-s}^g u)(x)$ and $T_t^g u$ is continuous on \overline{D} by Theorem 4.7. To prove that u is continuous on \overline{D} it is sufficient to show that the second term on the right-hand side tends to zero uniformly in x as $t \rightarrow 0$. This is, however, clear since we may estimate

$$\sup_{x \in \overline{D}} \left\{ \mathbb{E}_x \int_0^t e_g(s) f(\mathbf{X}_s) dL_s \right\} \leq z^{-1} \max_{l=1, \dots, N} \{U_l\} \sup_{x \in \overline{D}} \{\mathbb{E}_x L_t\},$$

where the right-hand side tends to zero as $t \rightarrow 0$ by Lemma 4.4.

It remains to show that u is given by (22). Note first that the *gauge function* $\mathbb{E}_x \int_0^\infty e_g(t) dL_t$ is \mathbb{P}_x -a.s. bounded for every $x \in \overline{D}$, hence the expression (22) is well-defined. By the Lax-Milgram Theorem there exists a weak solution of the boundary value problem (1), (4) such that for every $v \in H^1(D)$

$$\mathcal{E}(u, v) = \int_{\partial D} f(x)v(x) d\sigma(x) - \int_{\partial D} g(x)u(x)v(x) d\sigma(x).$$

By standard theory of linear elliptic boundary value problems u is bounded, cf. [29], which implies by [19, Theorem 5.4.2] together with the Fukushima decomposition (10) that for q.e. $x \in \overline{D}$, \mathbb{P}_x -a.s.

$$\tilde{u}(\mathbf{X}_t) = \tilde{u}(x) + \int_0^t \nabla \tilde{u}(\mathbf{X}_s) dM_s - \int_0^t f(\mathbf{X}_s) dL_s + \int_0^t g(\mathbf{X}_s) \tilde{u}(\mathbf{X}_s) dL_s.$$

Note that the second term on the right-hand side is a local martingale with respect to $\{\mathcal{F}_t, t \geq 0\}$ and that e_g is continuous, adapted to $\{\mathcal{F}_t, t \geq 0\}$ and of bounded variation. Multiplication by such functions leaves the class of local martingales invariant. Using integration by parts we obtain thus for q.e. $x \in \overline{D}$ and $t \geq 0$ the identity

$$\tilde{u}(\mathbf{X}_t) e_g(t) = \tilde{u}(x) + \int_0^t e_g(s) \nabla \tilde{u}(\mathbf{X}_s) dM_s - \int_0^t e_g(s) f(\mathbf{X}_s) dL_s,$$

where the second summand on the right-hand side is a local martingale. That is, there exists an increasing sequence $(\tau_k)_{k \in \mathbb{N}}$ of stopping times which tend to infinity such that for every $k \in \mathbb{N}$

$$\left\{ \int_0^{t \wedge \tau_k} e_g(s) \nabla \tilde{u}(\mathbf{X}_s) dM_s, t \geq 0 \right\}$$

is a martingale with respect to $\{\mathcal{F}_t, t \geq 0\}$. In particular we have for q.e. $x \in \overline{D}$ and every $k \in \mathbb{N}$

$$\tilde{u}(x) = \mathbb{E}_x \int_0^{t \wedge \tau_k} e_g(s) f(\mathbf{X}_s) dL_s + \mathbb{E}_x \tilde{u}(\mathbf{X}_{t \wedge \tau_k}) e_g(t \wedge \tau_k) \quad \text{for all } t \geq 0.$$

By the uniform integrability of $\{e_g(s), 0 \leq s \leq t\}$ with respect to \mathbb{P}_x , $x \in \overline{D}$, $t > 0$, together with the Monotone Convergence Theorem we obtain

$$\tilde{u}(x) = \mathbb{E}_x \int_0^t e_g(s) f(\mathbf{X}_s) dL_s + \mathbb{E}_x \tilde{u}(\mathbf{X}_t) e_g(t) \quad \text{for q.e. } x \in \overline{D}.$$

Letting $t \rightarrow \infty$ finally yields

$$\tilde{u}(x) = \mathbb{E}_x \int_0^\infty e_g(t) f(\mathbf{X}_t) dL_t \quad \text{for q.e. } x \in \overline{D},$$

where we have used the fact that \tilde{u} is bounded. As we have shown that the right-hand side in the last equality is continuous up to the boundary, the function u coincides with its quasi-continuous version \tilde{u} and the assertion holds for every $x \in \overline{D}$. \square

Remark 3. Note that the technique we used to prove Theorem 4.6 fails for the Neumann problem corresponding to the continuum model. This comes from the fact that in this case the gauge function becomes infinite. For the same reason Theorem 1.2 from [10], specialized to a zero lower-order term, does not yield the desired Feynman-Kac type formula for the continuum model either.

We can now generalize the martingale characterization obtained in [36] for weak solutions of (1), (4).

Theorem 4.8. *Suppose the conditions of Theorem 4.6 are satisfied. Then the following statements are equivalent:*

- (i) *u is the weak solution of the boundary value problem (1), (4)*
- (ii) *For every $x \in \overline{D}$ the expression*

$$(27) \quad \mathcal{M}_t(u) := u(\mathbf{X}_t) - u(x) - \int_0^t g(\mathbf{X}_s)u(\mathbf{X}_s) dL_s + \int_0^t f(\mathbf{X}_t) dL_s$$

is a continuous martingale with respect to $\{\mathcal{F}_t, t \geq 0\}$.

Proof. Let us assume that (i) holds. First recall that the integral with respect to L is defined in pathwise Lebesgue-Stieltjes sense with respect to the induced random measure $\lambda((s, t]) := L_t - L_s$ on $\mathbb{R}_+ \cup \{0\}$, which is absolutely continuous in the sense that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_B |\phi| d\lambda \leq \varepsilon$ holds for all measurable sets B with $\lambda(B) < \delta$ and bounded measurable functions ϕ . In particular this implies the continuity of the maps $t \mapsto \int_0^t f dL_s$ and $t \mapsto \int_0^t gu dL_s$, respectively. The solution u is continuous up to the boundary by Theorem 4.6. By the Markov property of \mathbf{X} we have for all $s \leq t$:

$$\begin{aligned} \mathbb{E}_x \{ \mathcal{M}_t(u) \} &= \mathbb{E}_{\mathbf{X}_s} \left\{ u(\mathbf{X}_{t-s}) + \int_0^{t-s} (f(\mathbf{X}_r) - g(\mathbf{X}_r)u(\mathbf{X}_r)) dL_r \right\} \\ &\quad + \int_0^s (f(\mathbf{X}_r) - g(\mathbf{X}_r)u(\mathbf{X}_r)) dL_r - u(x) \\ &= \mathbb{E}_{\mathbf{X}_s} \mathcal{M}_{t-s}(u) + \mathcal{M}_s(u) \end{aligned}$$

Thus in order to obtain (ii) it suffices to show that $\mathbb{E}_x \mathcal{M}_t(u) = 0$ \mathbb{P}_x -a.s. for all $t \geq 0$ and all $x \in \overline{D}$. From standard theory of strongly continuous semigroups it is known that $\mathbb{E}_x u(\mathbf{X}_t) = T_t u(x)$, considered as a Banach space valued mapping from \mathbb{R}_+ to $H^1(D)$, is continuously differentiable for every $t > 0$, cf. [37]. Note that we may write $p(s, x, y) = T_{s-t} p(t, y, x)$ for every $s \geq t > 0$ so that p , considered as a Banach space valued mapping is also continuously differentiable with derivative $\frac{d}{ds} p(s, x, y) = -\mathcal{L}_y p(s, x, y)$. We may thus differentiate the expression $\mathbb{E}_x u(\mathbf{X}_s)$ under the integral sign to obtain

$$\begin{aligned} \frac{d}{ds} \mathbb{E}_x u(\mathbf{X}_s) &= \int_D u(y) \frac{d}{ds} p(s, x, y) dy = - \int_{\partial D} p(s, x, y) f(y) d\sigma(y) \\ &\quad + \int_{\partial D} p(s, x, y) g(y) u(y) d\sigma(y), \end{aligned}$$

where we have used (i) together with the fact that for fixed $s > 0$ and $x \in \overline{D}$ the function $p(s, x, \cdot)$ belongs to $H^1(D)$. By integration from 0 to t together with Lemma 4.4 we arrive at

$$\mathbb{E}_x u(\mathbf{X}_t) - u(x) = \mathbb{E}_x \left\{ \int_0^t (-f(\mathbf{X}_s) + g(\mathbf{X}_s)u(\mathbf{X}_s)) dL_s \right\}$$

and (ii) is proved.

Now let us assume that (ii) holds. By the continuity of u and uniqueness of the Doob-Meyer decomposition the term $u(\mathbf{X}_t)$ is a continuous semimartingale with respect to $\{\mathcal{F}_t, t \geq 0\}$. Moreover $e_g(t)$ is continuous, adapted to $\{\mathcal{F}_t, t \geq 0\}$ and of bounded variation. Multiplication by such functions leaves the class of semimartingales invariant, i.e., $e_g(t)u(\mathbf{X}_t)$, $t \geq 0$ is a continuous semimartingale as well. In particular we may

define the Itô stochastic integral with respect to this semimartingale and integration from 0 to t of the expression $d(e_g(s)u(\mathbf{X}_s)) + \exp(-\int_0^s g(\mathbf{X}_r)f(\mathbf{X}_s)dL_s)$ yields another martingale with respect to $\{\mathcal{F}_t, t \geq 0\}$, namely

$$e_g(t)u(\mathbf{X}_t) - u(x) + \int_0^t e_g(s)f(\mathbf{X}_s)dL_s.$$

Let v denote the unique solution to (1), (4), then we know from the proof of Theorem 4.6 that

$$e_g(t)v(\mathbf{X}_t) - v(x) + \int_0^t e_g(s)f(\mathbf{X}_s)dL_s$$

is a martingale with respect to $\{\mathcal{F}_t, t \geq 0\}$. Hence $e_g(t)(u(\mathbf{X}_t) - v(\mathbf{X}_t))$ is a martingale with respect to $\{\mathcal{F}_t, t \geq 0\}$ and by taking the expectation we obtain $u(x) - v(x) = T_t^g(u - v)(x)$. That is, $\mathcal{E}^g(u - v, w) = 0$ for every $w \in H^1(D)$ or equivalently

$$\mathcal{E}^g(u, w) = \mathcal{E}^g(v, w) = \int_{\partial D} f(x)w(x)d\sigma(x) \quad \text{for every } w \in H^1(D)$$

which is statement (i). \square

5. PROBABILISTIC INTERPRETATION OF THE INVERSE CONDUCTIVITY PROBLEM

The inverse conductivity problem for the continuum model, the so-called *Calderón problem* reads as follows: *Given the Cauchy data on the boundary, i.e., all pairs of voltage and current patterns $(\phi, \partial_{\kappa\nu}u)$, each pair corresponding to a solution of the conductivity equation (1) with $u|_{\partial D} = \phi$, is it possible to determine the conductivity κ uniquely?*

Since we assume that D is a Lipschitz domain and the conductivity is uniformly elliptic and bounded, the solution of the Dirichlet boundary value problem is unique. Therefore, the Cauchy data can be described as a Dirichlet-to-Neumann map

$$\Lambda_\kappa: \phi \mapsto \psi = \partial_{\kappa\nu}u, \quad H^{1/2}(\partial D) \rightarrow H^{-1/2}(\partial D)$$

where both the domain and the range are given by the standard Trace Theorem. This means that the Calderón problem can be restated as *given Λ_κ , is it possible to determine κ uniquely?*

We have already demonstrated that solving the forward problem for the conductivity equation is intimately connected with the diffusion process \mathbf{X} . We start with the reflecting diffusion \mathbf{X} and we stop it at the first exit time τ_D from the domain D , leading to the representation of the solution as

$$u(x) = \mathbb{E}_x\phi(\mathbf{X}_{\tau_D}), \quad x \in \overline{D},$$

cf. Remark 2. Therefore, the forward problem related to the conductivity equation could be probabilistically interpreted as *given \mathbf{X} and the boundary data ϕ determine the corresponding potential u .*

Another way of thinking of the forward problem would be to just recover Λ_κ given κ since this is the actual inverse of the inverse problem. Since $\mathcal{L} = \nabla \cdot \kappa \nabla$ is the infinitesimal generator of the Markov process \mathbf{X} , we are tempted to seek for a Markov process $\widehat{\mathbf{X}}$ with the generator Λ_κ . This observation was first made by Hsu in 1986 for the reflecting Brownian motion [26]. The Dirichlet-to-Neumann map generates the so-called *boundary process* $\widehat{\mathbf{X}}$ associated with the Markov process \mathbf{X} , which we shall define below. This way the probabilistic interpretation of the forward problem could be stated as *given \mathbf{X} determine the associated boundary process $\widehat{\mathbf{X}}$.*

This leads to the following probabilistic interpretation of the Calderón problem: *Given a boundary process $\widehat{\mathbf{X}}$, is it possible to uniquely determine a process \mathbf{X} whose boundary process $\widehat{\mathbf{X}}$ is?*

Let us now show that this interpretation can be carried out rigorously in our setting. The boundary local time L is a nondecreasing, adaptive process that increases only when \mathbf{X} is on the boundary. Following [19], we define the right-continuous right-inverse τ of L by

$$(28) \quad \tau(s) := \sup\{r \geq 0 : L_r \leq s\}.$$

The random variable $\tau(s)$, $s \in [0, \infty)$, is a stopping time with respect to the right-continuous history (\mathcal{F}_t) of \mathbf{X} since $\{\tau(s) \geq t\} = \{L_t \leq s\} \in \mathcal{F}_t$ and moreover, by continuity of the sample paths of \mathbf{X} we see that for every $s \in [0, \infty)$, the process \mathbf{X} is on the boundary ∂D at time $\tau(s)$.

Definition 5.1. We define the *boundary process* $\widehat{\mathbf{X}}$ associated with \mathbf{X} as the time-changed trace

$$\widehat{\mathbf{X}}_t := \mathbf{X}_{\tau(t)}$$

and the *boundary filtration*

$$\widehat{\mathcal{F}}_t := \mathcal{F}_{\tau(t)}.$$

We know that the boundary local time L is a PCAF in the strict sense by Lemma 4.3 and therefore, the boundary ∂D is the quasi support of L , cf. [19, Theorem 5.1.5]. Moreover, since we have a Lipschitz domain, every boundary point is a regular point, and therefore, the boundary process is a Hunt process on the boundary ∂D , cf. [19, Theorem A.2.12., Theorem 6.2.1].

In the sequel, we will denote the transition semigroup of the boundary process by $\{\widehat{T}_t, t \geq 0\}$ and the generator of the semigroup by $\widehat{\mathcal{L}}$.

We note that the representation Theorem 4.5 can be expressed with the help of the boundary process $\widehat{\mathbf{X}}$, the first exit time τ_D and the first exit place X_{τ_D} .

Lemma 5.2. *Suppose the conditions of Theorem 4.5 are satisfied. Then the solution has a representation*

$$u(x) = \lim_{t \rightarrow \infty} \mathbb{E}_x v(X_{\tau_D}, t)$$

where

$$v(x, t) := \int_0^t \mathbb{E}_x f(\widehat{\mathbf{X}}_s) ds \quad \text{for all } x \in \overline{D}.$$

Proof. This follows from Theorem 4.5 by the strong Markov property and the change of variables formula

$$(29) \quad \int_{\tau(a)}^{\tau(b)} f(s) dL_s = \int_a^b f(\tau(s)) ds$$

which follows by Monotone Class Theorem from the observation that

$$[L_a, L_b] = \tau \circ g_{a,b},$$

where we have set $g_{a,b}(t) := [t \in [a, b]]$. □

We will next verify the observation of Hsu [26] for our setting.

Theorem 5.3. *Suppose assumption (A2) holds. The infinitesimal generator of the boundary process $\widehat{\mathbf{X}}$ coincides with the Dirichlet-to-Neumann map Λ_κ on $H^{3/2}(\partial D)$.*

Proof. When $\phi \in H^{3/2}(\partial D)$, the conductivity equation (1) with Dirichlet boundary value ϕ admits a solution in $H^2(D)$. We may apply the Fukushima decomposition

$$\tilde{u}(\mathbf{X}_t) = \tilde{u}(\mathbf{X}_0) + M_t^{[u]} + N_t^{[u]}$$

for all $t > 0$ and all $x \in \partial D$ since the Revuz measures corresponding to $M^{[u]}$ and $N^{[u]}$ are smooth measures of \mathbf{X} when $u \in H^2(D)$ and $\partial_{\kappa\nu}u \in H^{1/2}(\partial D)$. We may hence assume that u itself is quasi-continuous, i.e., $u = \tilde{u}$. Since u solves the conductivity equation, we have

$$N_t^{[u]} = \int_0^t \nabla \cdot (\kappa \nabla u)(\mathbf{X}_s) ds - \int_0^t \partial_{\kappa\nu}u(\mathbf{X}_s) dL_s = - \int_0^t \partial_{\kappa\nu}u(\mathbf{X}_s) dL_s.$$

Since $M^{[u]}$ is a martingale AF, we obtain that the process

$$Z_t = u(\mathbf{X}_t) - u(\mathbf{X}_0) + \int_0^t \partial_{\kappa\nu}u(\mathbf{X}_s) dL_s$$

is a martingale. Since $\tau(t)$ is a stopping time, we may apply the Optional Stopping Theorem and we obtain

$$0 = \mathbb{E}_x Z_{\tau(t)} = \mathbb{E}_x \phi(\widehat{\mathbf{X}}_t) - \phi(x) + \mathbb{E}_x \int_0^{\tau(t)} \partial_{\kappa\nu}u(\mathbf{X}_s) dL_s$$

for every $x \in \partial D$. After an application of the change of variables formula (29) we arrive to

$$(30) \quad \mathbb{E}_x \phi(\widehat{\mathbf{X}}_t) = \phi(x) - \mathbb{E}_x \int_0^t \partial_{\kappa\nu}u(\widehat{\mathbf{X}}_s) ds = \phi(x) - \mathbb{E}_x \int_0^t \Lambda_\kappa \phi(\widehat{\mathbf{X}}_s) ds.$$

The identity (30) applied to $\widehat{T}_r \phi$ gives

$$\widehat{T}_{t+r} \phi = \widehat{T}_t(\widehat{T}_r \phi) = \widehat{T}_r \phi - \int_0^t \widehat{T}_s \Lambda_\kappa \widehat{T}_r \phi ds.$$

Therefore, the function $v(r) = \widehat{T}_r \phi$ solves the abstract Cauchy problem (11) with both Λ_κ and the generator $\widehat{\mathcal{L}}$ of the boundary semigroup in place of \mathcal{L} in (11) for the initial value ϕ . This proves the claim. \square

Remark 4. In this section we will not try to obtain the optimal regularity assumptions for the conductivity κ . Neither will we try to analyze the optimal regularity needed for the domain D . For instance, the assumption (A2) in Theorem 5.3 is clearly not optimal. The assumption is needed for the elliptic regularity so that we have a simple representation for the additive functional $N^{[u]}$ with zero energy. This could be improved by using an approximation procedure similar to the one we used in the proof of Theorem 4.5. However, we will leave these improvements for future work.

We can now elaborate the probabilistic inverse problem a bit further. For every given κ we have an associated diffusion process \mathbf{X}_κ . The above construction associates the diffusion process \mathbf{X}_κ with its boundary process $\widehat{\mathbf{X}}_\kappa$. Suppose we are given a boundary process $\widehat{\mathbf{X}}$ and we know a priori that there exists at least one κ_0 such that $\widehat{\mathbf{X}} = \widehat{\mathbf{X}}_{\kappa_0}$. Note that equality in the sequel means equality in distribution. The uniqueness question related to the Calderón problem would then be recasted as *suppose there exists a κ such that $\widehat{\mathbf{X}} = \widehat{\mathbf{X}}_\kappa$. Does it follow that $\mathbf{X}_\kappa = \mathbf{X}_{\kappa_0}$?* The reconstruction problem can be stated as *reconstruct the process \mathbf{X} such that $\widehat{\mathbf{X}}_\kappa = \widehat{\mathbf{X}}$.*

The Calderón problem in 2 dimensions is known to be solvable for isotropic $\kappa \in L^\infty(D)$. Given the boundary process $\widehat{\mathbf{X}} = \widehat{\mathbf{X}}_{\kappa_0}$ we can thus uniquely determine the generator $\Lambda = \Lambda_{\kappa_0}$. The celebrated result of Astala and Päivärinta [3] says that whenever $\Lambda_\kappa = \Lambda$ and both κ and κ_0 are isotropic, uniformly bounded and uniformly elliptic, then $\kappa = \kappa_0$. Therefore, the equality $\mathbf{X}_\kappa = \mathbf{X}_{\kappa_0}$ must hold as well.

The recent result by Haberman and Tataru [23] implies the same for higher dimensional cases when κ and κ_0 are assumed to be $C^1(D)$ or if they are Lipschitz continuous and close to identity in certain sense.

When the conductivity is not assumed to be isotropic the uniqueness has always an obstruction, namely we have $\Lambda_{\kappa_0} = \Lambda_{\kappa_1}$, whenever $\kappa_1 = F_*\kappa_0$ is the push-forward conductivity by a diffeomorphism F on D that leaves the boundary ∂D invariant. In the plane, this is known to be the only obstruction by the result of Astala, Lassas and Päivärinta [2] which holds without additional regularity assumptions on the conductivity. In higher dimensions the question is still very much open in general, see [12] for further discussion.

These results from analysis all rely on so-called *complex geometric optics solutions* and the authors are not aware of any probabilistic interpretation of these solutions. Therefore, a probabilistic solution to the (probabilistic interpretation of the) Calderón problem should use some other techniques.

One possible approach could be to understand the structure of the boundary process more thoroughly and attempt to “join the dots” by constructing the compatible excursions between the boundary points or by showing the uniqueness of the distribution of the compatible excursions. As a first step towards that direction we adapt the representation result from [26] to our setting. The following Theorem is the main result of this section.

Theorem 5.4. *Suppose both assumptions (A1) and (A2) hold. Then the Dirichlet-to-Neumann map Λ_κ is of form*

$$\Lambda_\kappa \phi = b \cdot \nabla_T \phi + A_0$$

where b is a vector field given by

$$b := \Lambda_\kappa \text{id}.$$

The operator A_0 is the integral operator

$$A_0 \phi(x) := 2 \int_{\partial D} A_1(x) \phi(y) N(x, y) \, d\sigma(y),$$

where $A_1(x) \phi(y) := \phi(y) - \phi(x) - \nabla_T \phi(x) \cdot (y - x)$ and N is the conormal derivative of the Poisson kernel, explicitly given by the transition density kernel p_0 of the killed diffusion \mathbf{X}_0 as

$$N(x, y) = \int_0^\infty \partial_{\kappa\nu(x)} \partial_{\kappa\nu(y)} p_0(t, x, y) \, dt.$$

From this representation, we see that the generator of the boundary process $\widehat{\mathbf{X}}$ is of form

$$\widehat{\mathcal{L}} = \nabla \cdot \widehat{A} \nabla + b \cdot \nabla + A_0,$$

with diffusion coefficient $\widehat{A} = 0$. This is an integro-differential operator in the sense of Lepeltier and Marchal [32]. As in Hsu [26], this means that $\widehat{\mathbf{X}}$ is a pure jump process without diffusion part and the jump distribution can be described with the help of [32, Théorème 10].

Lemma 5.5. *For every Borel measurable $\phi: \partial D \times \partial D \rightarrow \mathbb{R}_+$ vanishing on the diagonal and any stopping time τ for $\widehat{\mathbf{X}}$, we have*

$$\mathbb{E}_x \sum_{s \leq \tau} \phi(\widehat{\mathbf{X}}_{s-}, \widehat{\mathbf{X}}_s) [\widehat{\mathbf{X}}_{s-} \neq \widehat{\mathbf{X}}_s] = 2\mathbb{E}_x \int_0^\tau \int_{\partial D} \phi(\widehat{\mathbf{X}}_s, y) N(\widehat{\mathbf{X}}_s, y) d\sigma(y) ds.$$

Proof. Suppose first that $\text{diam}(D) \leq 1$. We note that the operator A_0 in Lemma 5.4 coincides with the integral operator causing the jumps, namely when $\phi \in H^{3/2}(\partial D)$ and it is continued as $\psi \in H^2(\mathbb{R}^d)$ so that ϕ and its tangential derivative are continued as constants along the conormal directions in the neighborhood of the boundary ∂D , then for every $x \in \partial D$ we have

$$A_0\phi(x) = \int_{\mathbb{R}^d \setminus \{0\}} (\psi(x+z) - \psi(x) - [|z| \leq 1]z \cdot \nabla\psi(x)) S(x, dz),$$

where we have set $S(x, dz) := 2N(x, x+z) d\sigma_x(z)$ and $\sigma_x(B) := \sigma(x+B)$ for every Borel set $B \subset \mathbb{R}^d$. In the same way we can extend the drift term and we obtain an extended process $\overline{\mathbf{X}}$ into whole space \mathbb{R}^d . Since we know that $\widehat{\mathbf{X}}_t \in \partial D$ for all $t \geq 0$, it follows that the extended process $\overline{\mathbf{X}}$ will stay on the boundary ∂D if we start it from the boundary and it coincides with $\widehat{\mathbf{X}}$ there.

For this extended process $\overline{\mathbf{X}}$ we can apply the result [32, Théorème 10] and we obtain

$$\begin{aligned} \mathbb{E}_x \sum_{s \leq \tau} \phi(\widehat{\mathbf{X}}_{s-}, \widehat{\mathbf{X}}_s) [\widehat{\mathbf{X}}_{s-} \neq \widehat{\mathbf{X}}_s] \\ = 2\mathbb{E}_x \int_0^\tau \int_{\mathbb{R}^d \setminus \{0\}} \phi(\widehat{\mathbf{X}}_s, \widehat{\mathbf{X}}_s + y) N(\widehat{\mathbf{X}}_s, \widehat{\mathbf{X}}_s + y) d\sigma_{\widehat{\mathbf{X}}_s}(y) ds. \end{aligned}$$

The claim follows now in this special case by change of integration variable.

The general case follows by scaling: Let us denote $\mathbf{X}_t^R := R^{-1}\mathbf{X}_t$. This is a reflecting diffusion process corresponding to κ^R on a domain D^R , where $D^R := R^{-1}D$ and $\kappa^R(x) := R^{-2}\kappa(Rx)$. Since the diameter of D^R is one, the claim holds for the boundary process $\widehat{\mathbf{X}}^R$ of \mathbf{X}^R .

Let L^R denote the local time of \mathbf{X}^R on the boundary ∂D^R . By definition, this is in Revuz correspondence with the surface measure σ^R of the boundary ∂D^R . By using the Revuz correspondence [19, Theorem 5.1.3] and change of variables, it follows that

$$L_t^R = RL_t.$$

Therefore, the right-inverse τ^R of the local time L^R has a scaling law

$$\tau^R(t) = \tau(R^{-1}t)$$

which in turn implies that the boundary processes scale by the law

$$\widehat{\mathbf{X}}_t^R = R^{-1}\widehat{\mathbf{X}}_{R^{-1}t}$$

and that η is an $\widehat{\mathbf{X}}$ -stopping time if and only if $R\eta$ is an $\widehat{\mathbf{X}}^R$ -stopping time.

If we denote by N^R , the conormal derivative of the Poisson kernel of \mathbf{X}^R and compute the scaling law, we find out that

$$N^R(x, y) = R^{d-2}N(Rx, Ry).$$

With all these scaling laws, we are now ready to prove the claim for $\widehat{\mathbf{X}}$. Let $\phi^R(x, y) := \phi(x, y)$ and let η be an $\widehat{\mathbf{X}}$ -stopping time. We have

$$\mathbb{E}_x \sum_{s \leq \eta} \phi(\widehat{\mathbf{X}}_s, \widehat{\mathbf{X}}_{s-}) [\widehat{\mathbf{X}}_s \neq \widehat{\mathbf{X}}_{s-}] = \widehat{\mathbb{E}}_{R^{-1}x} \sum_{s \leq \eta R} \phi^R(\widehat{\mathbf{X}}_s^R, \widehat{\mathbf{X}}_{s-}^R) [\widehat{\mathbf{X}}_s^R \neq \widehat{\mathbf{X}}_{s-}^R],$$

where $\widehat{\mathbb{E}}_{R^{-1}x}$ denotes the expectation given $\widehat{\mathbf{X}}_0^R = R^{-1}x$. By the first part of the proof, the right-hand side is

$$2\widehat{\mathbb{E}}_{R^{-1}x} \int_0^{\eta R} \int_{\partial D^R} \phi^R(\widehat{\mathbf{X}}_s^R, y) N^R(\widehat{\mathbf{X}}_s^R, y) d\sigma(y) ds.$$

With the change of variables $y' = Ry$ and $s' = sR^{-1}$ and the scaling law $N^R(\widehat{\mathbf{X}}_{Rs}^R, R^{-1}y) = R^{d-2}N(\widehat{\mathbf{X}}_s, y)$, the claim follows. \square

This result states that the pair $(\widetilde{N}, \text{id})$ is the Lévy system (see [19, Definition A.3.7]) of the Hunt process $\widehat{\mathbf{X}}$ where \widetilde{N} is the kernel on $(\partial D, \mathcal{B}(\partial D))$ given by

$$\widetilde{N}(x, B) := 2 \int_B [x \notin B] N(x, y) d\sigma(y)$$

Since the PCAF $\text{id}_t = t$ with respect to $\widehat{\mathbf{X}}$ has the Revuz measure σ , we see that $\frac{1}{2}\widetilde{N}(x, B) d\sigma(x)$ is the *jumping measure* J of the boundary process $\widehat{\mathbf{X}}$ (see [19, Theorem 5.3.1]).

The same proof as in [26, Proposition 4.4] shows that the random set of jump times $\{\widehat{\mathbf{X}}_{s-} \neq \widehat{\mathbf{X}}_s\}$ is a countable and dense set and that there is a constant $c > 0$, depending only on the domain D , such that after any given time $t \in \mathbb{R}_+$ there are always infinitely many jumps of size at least c .

For the proof of Theorem 5.4 we need the following auxiliary results.

Lemma 5.6. *Suppose assumption (A2) holds. Then the transition density kernel p_0 of the killed diffusion $\mathbf{X}_t^0 := [t < \tau_D]\mathbf{X}_t$ with lifetime τ_D is Hölder continuous with respect to (t, x, y) , is in $H_0^2(D)$ with respect to x and y and continuously differentiable with respect to t as a Banach space valued map.*

Proof. The Hölder continuity follows from Theorem 3.1 by Markov property, since

$$p_0(t, x, y) = p(t, x, y) - \mathbb{E}_x \{p(t - \tau_D, X_{\tau_D}, y) [t > \tau_D]\}.$$

Moreover, we can carry out the same construction as in the proof of Lemma 4.1 only by replacing the space $H^1(D)$ with $H_0^1(D)$. This yields an eigenfunction expansion of p_0 , namely

$$p_0(t, x, y) = \sum_{j=0}^{\infty} e^{-\mu_j t} \psi_j(x) \psi_j(y),$$

which can be seen to be in $H_0^2(D)$ with respect to both variables x and y by standard elliptic regularity theory, cf. [22]. \square

Lemma 5.7. *Suppose assumption (A2) holds, then for every bounded and measurable ϕ on ∂D and every bounded and measurable ψ on \mathbb{R}_+ we have*

$$\mathbb{E}_x \phi(\mathbf{X}_{\tau_D}) \psi(\tau_D) = - \int_{\partial D} \int_{\mathbb{R}_+} \phi(y) \psi(t) \partial_{\kappa \nu(y)} p_0(t, x, y) dt d\sigma(y).$$

Proof. The result follows by generalizing the results and proofs of [1, Theorem A.3.2., Lemma A.3.3.] using the transition density kernel p_0 of the killed diffusion \mathbf{X}^0 instead of the transition density kernel of the killed Brownian motion. The proofs and the claims generalize in a straightforward manner by replacing the Brownian motion by the diffusion \mathbf{X} , the harmonic functions h by the weak solutions of the conductivity equation $\nabla \cdot \kappa \nabla h = 0$ and the normal derivatives by conormal derivatives. The regularity we need for the proofs to go through follow from Lemma 5.6. \square

Lemma 5.8. *Suppose assumption (A1) holds and the boundary ∂D is of class $C^{1,1}$. For every $x \in \partial D$ the function $y \mapsto N(x, y)(|y - x|^2 \wedge 1)$ is integrable with respect to the surface measure σ .*

Proof. When $\kappa \equiv 1$, the proof given in [26] for $\kappa \equiv \frac{1}{2}$ generalizes and gives the claim for kernel N_1 corresponding to $\kappa \equiv 1$. If we assume (A1), then the operator $\Lambda_\kappa - \Lambda_1$ is a smoothing operator which follows by the standard elliptic regularity, cf. [24]. This implies that the kernels N_1 and N have the same leading singularities and the claim follows from the estimate for N_1 . \square

Proof of Theorem 5.4. Suppose $\phi \in C^2(\partial D) \cap H^{3/2}(\partial D)$. The solution of the Dirichlet problem is by Lemma 5.7

$$u(x) = \mathbb{E}_x \phi(\mathbf{X}_{\tau_D}) = \int_{\partial D} \int_{\mathbb{R}_+} \phi(y) \partial_{\kappa \nu(y)} p_0(t, x, y) dt d\sigma(y) =: \mathcal{K}\phi(x).$$

Therefore, the Dirichlet-to-Neumann map Λ_κ maps ϕ to

$$(31) \quad \Lambda_\kappa \phi(x) = \partial_{\kappa \nu(x)} \mathcal{K}\phi(x).$$

Let us extend ϕ and its first order tangential derivative $V := \nabla_T \phi$ into the neighborhood of the boundary as constant along the conormal directions. We will denote the extensions $\tilde{\phi}$ and \tilde{V} , respectively. We will compute the conormal derivative of the function

$$w := \mathcal{K}\phi - \tilde{\phi} \mathcal{K} \mathbf{1} - \sum_{j=1}^d \tilde{V}_j (\mathcal{K} W_j - \tilde{W}_j \mathcal{K} \mathbf{1}),$$

where $\{W\}$ is a vector field on the boundary defined by $W(y) := y_T$ as the projection to the tangent plane going through the point y . By construction the conormal derivative commutes with multiplication by the extended functions and vector fields. Therefore,

$$\partial_{\kappa \nu} w = \Lambda_\kappa \phi - \phi \Lambda_\kappa \mathbf{1} - \sum_{j=1}^d V_j \Lambda_\kappa W_j = \Lambda_\kappa \phi - V \cdot \Lambda_\kappa W,$$

where $\Lambda_\kappa \mathbf{1} = 0$ since $u(x) = 1$ in \overline{D} is the unique solution to the Dirichlet problem and therefore, the conormal derivative vanishes identically. We can compute the left-hand side in a different way, namely

$$\nabla w(x) = \nabla_x \int_{\partial D} (\phi(y) - \tilde{\phi}(x) - \tilde{V}(x) \cdot (\tilde{W}(y) - \tilde{W}(x))) K(x, y) d\sigma(y)$$

for almost every x in a neighborhood of boundary. By Lemma 5.8 we can use the Dominated Convergence Theorem to take the differentiation inside the integration and we obtain thus

$$\partial_{\kappa \nu} w(x) = \int_{\partial D} (\phi(y) - \phi(x) - \nabla_T \phi(x) \cdot (y - x)) N(x, y) d\sigma(y) = A_0 \phi(x)$$

for almost every $x \in \partial D$. \square

We have demonstrated that the boundary process $\widehat{\mathbf{X}}$ is intimately related with the Dirichlet-to-Neumann map Λ_κ . In order to solve the probabilistic inverse problem of recovering the process \mathbf{X}_κ inside the domain, we should provide the *excursions* between the points $\widehat{\mathbf{X}}_{t-}$ and $\widehat{\mathbf{X}}_t$ at the jumps so that the Lévy system of the boundary process $\widehat{\mathbf{X}}$ and *excursion law* of these excursions would be consistent, cf. [26]. These excursions could be regarded as point processes on the space of continuous functions that start at the boundary, stay inside the domain and stop at the boundary.

The point process of excursions for the case $\kappa \equiv 1$ was defined in [26] and under a certain consistency assumption it was shown that the interior RBM \mathbf{X} can be reconstructed from its point excursions and the boundary process. However, the consistency assumption was derived by using RBM to begin with and as it is noted in [26], there might be other consistency assumptions leading to other constructions. Showing that there are no other consistent constructions is equivalent to the probabilistic inverse problem.

However, we will not attempt to analyze the excursion processes for conductivities κ in this paper and therefore, we will leave the analysis of the probabilistic inverse problem for future work.

6. CONCLUSION

We have obtained probabilistic interpretations of both the direct as well as the inverse problem of electrical impedance tomography. Using the theory of symmetric Dirichlet spaces we have derived Feynman-Kac type representation formulae generalizing the probabilistic representations from [4, 7, 36]. These formulae are potentially relevant in statistical inversion theory as well as stochastic numerics of problems involving random, rapidly oscillating coefficients. Furthermore we have given a probabilistic formulation of Calderón's inverse conductivity problem, generalizing results from [26], which may yield a novel perspective and a probabilistic set of tools when it comes to studying the open question of unique determinability of merely measurable conductivities for $d \geq 3$.

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